C. H. SUH
raduate Student in Mechanical Engineering,
University of California, Berkeley, Calif.,
and Postgraduate Research Engineer,
Biomechanics Laboratory,
University of California Medical Center,
San Francisco, Calif. Assoc. Mem. ASME

C. W. RADCLIFFE
Professor of Mechanical Engineering,
University of California, Berkeley, Calif.,
and Research Engineer,
Biomechanics Laboratory,
University of California Medical Center,
San Francisco, Calif. Assoc. Mem. ASME

Synthesis of Plane Linkages With Use of the Displacement Matrix

A generalized matrix for the description of rigid body displacement in two dimensions is developed. This displacement matrix is applied to the synthesis of plane linkages used for rigid body guidance, path generation, and function generation.

1 Introduction

BEGINNING in 1955 with the work of F. Freudenstein ıd his co-workers [1, 2],[1] there has been an extensive developent of analytical methods for the synthesis of plane mechanism hich make efficient use of the capabilities of modern digital mputers. Major emphasis has been placed on the use of comex polar vector notation to describe relative positions in the ıkage of an assumed type from which analytical expressions for lative displacements, velocities, or accelerations can be dermined. These equations are then combined with design contions to give a set of simultaneous equations to be solved for known mechanism parameters.

A second and somewhat different design method is the synthesis linkages directly from specified finite displacements given in merical form. Wilson [3] has developed design equations ich locate center-point and circle-point curves of plane mechams in terms of the classic rotation matrix operator combined th the displacement of a point in the moving body initially ated at the origin of the coordinate system. The rotation trix is somewhat limited in its application since all rotations ıst be specified about axes passing through the origin of the rdinate system.

The present paper gives an extension of the finite displacement thod for plane mechanism synthesis using a 3 × 3 displacent matrix operator, which allows a more generalized descrip- of plane displacement than the usual 2 × 2 plane rotation trix. A future paper will describe the application of the 4 × 4 placement matrix operator to problems in the synthesis of ee-dimensional linkages.

Geometric Transformations

Geometric transformations [4] are a part of the mathematical ion of *function*. The simplest type of geometric transforma- is the *point transformation*, in which every point considered an element of one space is transformed into a corresponding nt in a second space. A particularly simple group of geometric nsformations, useful in kinematics, is the *affine transformation*, which points located on a straight line in one space are transned into corresponding points on a straight line in a second ce.

An n-dimensional affine space A^n is a set of elements (points) having a one-to-one mapping onto the n-dimensional vector space V^n. An affine transformation in two-dimensional space is defined analytically when x_2, y_2 are linear functions of x_1, y_1. Expressed as a homogeneous matrix equation,

$$\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} \tag{1}$$

Rigid body displacement without distortion can be considered as a special case of an affine transformation.

3 Displacement: Mathematical Description

A particular rigid body displacement, as defined above, can be considered as one element of the group which consists of the system $\mathfrak{D}$ of all rigid body displacements. We let $[D]$ in matrix form represent an element of the system $\mathfrak{D}$ and adopt matrix multiplication as the rule of combination of the group.

From the mathematical definition of a group [5] we may state the following conditions which must be satisfied by displacement matrices:

(*a*) The product of any two of the system of generalized displacement matrices is also a displacement matrix which forms an element of the same system in Euclidian space.

$$[D_1][D_2] = [D] \tag{2}$$

(*b*) The product of any three displacement matrices is associative.

$$([D_1][D_2])[D_3] = [D_1]([D_2][D_3]) \tag{3}$$

(*c*) There exists an identity matrix $[I]$ which is an element of the system.

(*d*) For any displacement matrix $[D_1]$ in the system, there exists an inverse $[D_1]^{-1}$ in the same system such that

$$[D_1][D_1]^{-1} = [I] \tag{4}$$

4 Basic Displacement Matrices

After establishment of the mathematical and geometric basis for the general two-dimensional displacement matrix, it is important to examine the analytical form of the displacement matrix under certain conditions. Displacement matrices will be formulated as combinations of translation and rotation about the origin of the coordinate system, in order to display the components of

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[1] Numbers in brackets designate References at end of paper.

ontributed by the Machine Design Division and presented at Mechanisms Conference, Lafayette, Ind., October 10–12, 1966, of AMERICAN SOCIETY OF MECHANICAL ENGINEERS. Manuscript ived at ASME Headquarters, June 29, 1966. Paper No. 66—h-19.

the matrix analytically in a form which will allow the resolution of a numerical matrix into its components.

ranslation in Two Dimensions

Refer to Fig. 1, which indicates a translation displacement of a lane on a plane where each point in the plane undergoes a dislacement Δx, Δy. Under these conditions

$$[D_{12}]_T = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \qquad (5)$$

here

$$\Delta x = x_2 - x_1, \qquad \Delta y = y_2 - y_1$$

otation in Two Dimensions About the Origin

Fig. 2 illustrates the rotation of a plane containing a point P oout a fixed origin of the coordinate system. The rotation matrix kes the familiar form,

$$[D_{12}]_R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (6)$$

eneral Motion in Two Dimensions

The displacement matrix for general plane motion can be rmulated in several ways. Assume a rotation pole $A_1(x_0, y_0)$ ists which forms a center of rotation in the plane. First, form displacement matrix $[D_1]_T$ such that the pole $A_1(x_0, y_0)$ will incide with the origin.

$$[D_1]_T = \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (7)$$

Since

$$\Delta x = 0 - x_0 = -x_0, \qquad \Delta y = 0 - y_0 = -y_0.$$

cond, allow the rotation displacement $[D_2]_R$ about the origin 0)

$$[D_2]_R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (8)$$

ally, translate the point A_1 back to its original position.

$$[D_3]_{-T} = \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (9)$$

erefore,

$$] = [D_3]_{-T}[D_2]_R[D_1]_T$$

$$= \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[D_{12}] = \begin{bmatrix} \cos\theta & -\sin\theta & x_0(1 - \cos\theta) + y_0 \sin\theta) \\ \sin\theta & \cos\theta & y_0(1 - \cos\theta) - x_0 \sin\theta) \\ 0 & 0 & 1 \end{bmatrix} \qquad (10)$$

ig. 3 illustrates two-dimensional motion in which the position he rotation pole is unknown. First rotate $\overline{A_1B_1}$ about the n through the angle θ to position $\overline{A_1'B_1'}$. The line $\overline{A_1'B_1'}$ is moved to position $\overline{A_2B_2}$ by direct translation.

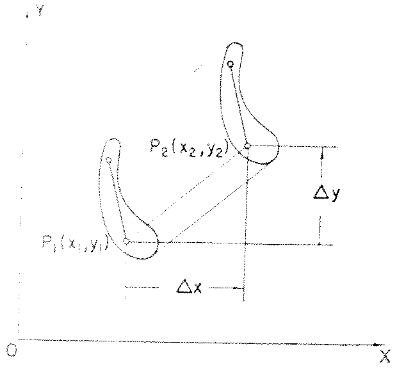

Fig. 1 Two-dimensional translation

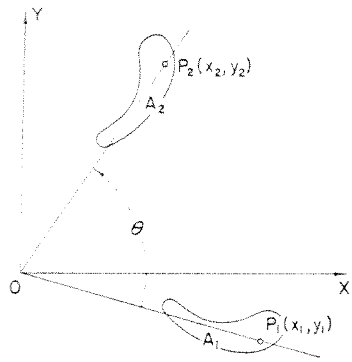

Fig. 2 Rotation in two dimensions about the origin

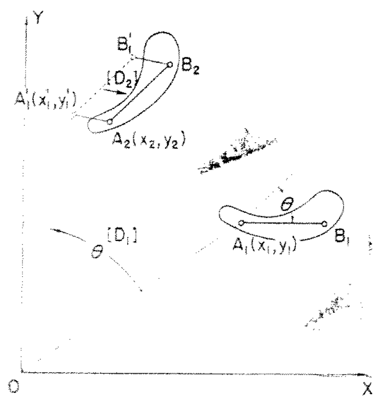

Fig. 3 General two-dimensional displacement

$$[D_{12}] = [D_2]_T[D_1]_R$$

$$= \begin{bmatrix} 1 & 0 & (x_2 - x_1') \\ 0 & 1 & (y_2 - y_1') \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[D_{12}] = \begin{bmatrix} \cos\theta & -\sin\theta & (x_2 - x_1') \\ \sin\theta & \cos\theta & (y_2 - y_1') \\ 0 & 0 & 1 \end{bmatrix}$$

Since

$$A_1' = \begin{bmatrix} x_1' \\ y_1' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \cos\theta - y_1 \sin\theta \\ x_1 \sin\theta + y_1 \cos\theta \\ 1 \end{bmatrix}$$

we obtain

$$[D_{12}] = \begin{bmatrix} \cos\theta & -\sin\theta & (x_2 - x_1 \cos\theta + y_1 \sin\theta) \\ \sin\theta & \cos\theta & (y_2 - x_1 \sin\theta - y_1 \cos\theta) \\ 0 & 0 & 1 \end{bmatrix} \qquad (11)$$

Comparing (10) and (11), we note that the rotation pole $P(x_0, y_0)$ may be determined from $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, and θ by equating corresponding elements of the displacement matrix.

A third form would be expressed in terms of the coordinates of specific points of the moving plane in the initial and final positions. Three points, A, B, and C, are specified in positions $A_1(x_1, y_1)$, $B_1(x_1', y_1')$, and $C_1(x_1'', y_1'')$ and in positions $A_2(x_2, y_2)$, $B_2(x_2', y_2')$, and $C_2(x_2'', y_2'')$. The $[D_{12}]$ matrix may be found by noting that

$$\begin{bmatrix} x_2 & x_2' & x_2'' \\ y_2 & y_2' & y_2'' \\ 1 & 1 & 1 \end{bmatrix} = [D_{12}] \begin{bmatrix} x_1 & x_1' & x_1'' \\ y_1 & y_1' & y_1'' \\ 1 & 1 & 1 \end{bmatrix}$$

Therefore

$$[D_{12}] = \begin{bmatrix} x_2 & x_2' & x_2'' \\ y_2 & y_2' & y_2'' \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_1' & x_1'' \\ y_1 & y_1' & y_1'' \\ 1 & 1 & 1 \end{bmatrix}^{-1} \quad (12)$$

It is often convenient to form the displacement matrix from the change in position of two points only in the rigid body. In this case the coordinates of a third point in the body may be found (and used as x'', y'') by rotating (x', y') 90 deg about (x, y). This leads to a fourth expression similar to equation (12) but involving two moving points only.

$$[D_{12}] = \begin{bmatrix} x_2 & x_2' & (-y_2' + x_2 + y_2) \\ y_2 & y_2' & (x_2' + y_2 - x_2) \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_1' & (-y_1' + x_1 + y_1) \\ y_1 & y_1' & (x_1' + y_1 - x_1) \\ 1 & 1 & 1 \end{bmatrix}^{-1} \quad (13)$$

Any of the forms given in equations (10), (11), (12), and (13) will result in a 3 × 3 numerical matrix of the form

$$[D_{12}] = \left[\begin{array}{cc|c} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ \hline 0 & 0 & 1 \end{array}\right] \quad (14)$$

The matrix equation

$$\begin{bmatrix} X_2 \\ Y_2 \\ 1 \end{bmatrix} = [D_{12}] \begin{bmatrix} X_1 \\ Y_1 \\ 1 \end{bmatrix} \quad (15)$$

represents a coordinate transformation in which any point of the set which comprises the moving plane is transformed from position 1 into position 2.

5 General Method of Synthesis

The basic problem in the synthesis of plane linkages is to locate those points of the moving plane which, as the plane assumes specified positions, assume a series of positions that lie on a circular arc. These particular points, designated circle points, can be used as hinge points in the moving plane. Two links, each with one end connected to the moving hinge points (circle points) and the other end to ground at the center of the corresponding circular arc (center points), will guide the plane through the specified positions.

Assume that a plane displacement is characterized in the general form of a $[D]$ matrix similar to equation (11).

Let X_1 and Y_1 be the unknown coordinates of a moving pivot in position 1. The nth position of the moving pivot is expressed in terms of the first position as

$$\begin{bmatrix} X_n \\ Y_n \\ 1 \end{bmatrix} = [D_{1n}] \begin{bmatrix} X_1 \\ Y_1 \\ 1 \end{bmatrix} \qquad n = 2, 3, \ldots \quad (16)$$

If (X_0, Y_0) are the coordinates of the corresponding fixed pivot, the condition of constant link length requires that

$$(X_1 - X_0)^2 + (Y_1 - Y_0)^2 = (X_n - X_0)^2 + (Y_n - Y_0)^2 \qquad n = 2, 3, \ldots \quad (17)$$

Squaring expressions for X_n and Y_n found from equation (16) and collecting terms leads to

$$X_1[A_{13n} \cos\theta_{1n} + A_{23n} \sin\theta_{1n} - X_0 \cos\theta_{1n} - Y_0 \sin\theta_{1n} + X_0] + Y_1[A_{23n} \cos\theta_{1n} - A_{13n} \sin\theta_{1n} + X_0 \sin\theta_{1n} - Y_0 \cos\theta_{1n} + Y_0] = A_{13n}X_0 + A_{23n}Y_0 - {}^1/_2(A_{13n}^2 + A_{23n}^2) \qquad n = 2, 3, \ldots \quad (18)$$

where

$$A_{13n} = x_n - x_1 \cos\theta_{1n} + y_1 \sin\theta_{1n}$$

$$A_{23n} = y_n - x_1 \sin\theta_{1n} - y_1 \cos\theta_{1n}$$

In equation (18) there are four unknowns, X_0, Y_0, X_1, and Y_1. For a three-position guidance problem, two equations (18) would be available. Therefore, any two of the unknowns could be specified arbitrarily and the equations used to solve for the remaining pair of coordinates. A double infinity of possible solutions is theoretically possible.

If a fourth position is specified, one coordinate may be specified arbitrarily and the set of three equations solved for coordinates of points on both the center-point (X_0, Y_0) and circle-point (X_1, Y_1) curves.

Specification of a fifth position results in the possibility of, but not necessarily assurance of, a unique four-bar guiding linkage.

If it is desired to guide the moving member by use of a slider-crank mechanism, a straight-line equation is used in place of the circle equation given in equation (17). The equation of a straight line passing through three points $P_1(X_1, Y_1)$, $P_2(X_2, Y_2)$, $P_3(X_3, Y_3)$ may be expressed conveniently in the form,

$$\begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix} = 0 \quad (19)$$

The displacement matrices $[D_{12}]$ and $[D_{13}]$ may be used to express $P_2(X_2, Y_2)$ and $P_3(X_3, Y_3)$ in terms of $P_1(X_1, Y_1)$. This leads to a single equation with two unknowns X_1, Y_1. Either of the unknown coordinates may be assumed arbitrarily and the second is determined from the equation. An infinite number of possible slider positions is theoretically possible in the three-position guidance problem.

When a fourth position of the moving plane is specified, there are two straight-line equations which when solved simultaneously may result in a unique position for the slider in the first position.

The direction of motion of the slider is easily found from the equation for the slope as given below.

$$\theta = \tan^{-1} \frac{(Y_2 - Y_1)}{(X_2 - X_1)} \quad (20)$$

6 The Three-Position Guidance Problem

Example Problem 1

Three positions of a moving member are specified (as shown in Fig. 4), for which it would be difficult to design a guiding four-bar linkage by the use of the pole triangle or equivalent methods.

$$A_1 = A_1(x_1, y_1) = (1, 1)$$

$$A_2 = A_2(x_2, y_2) = (2, 0.5) \qquad \theta_{12} = 0 \text{ deg}$$

$$A_3 = A_3(x_3, y_3) = (3, 1.5) \qquad \theta_{13} = 45 \text{ deg}$$

A four-bar linkage will be designed with the fixed pivot for one crank located at $B_0(X_0, Y_0) = B_0(0, 0)$ and the fixed pivot for the second crank located at $C_0(X_0', Y_0') = C_0(5, 0)$.

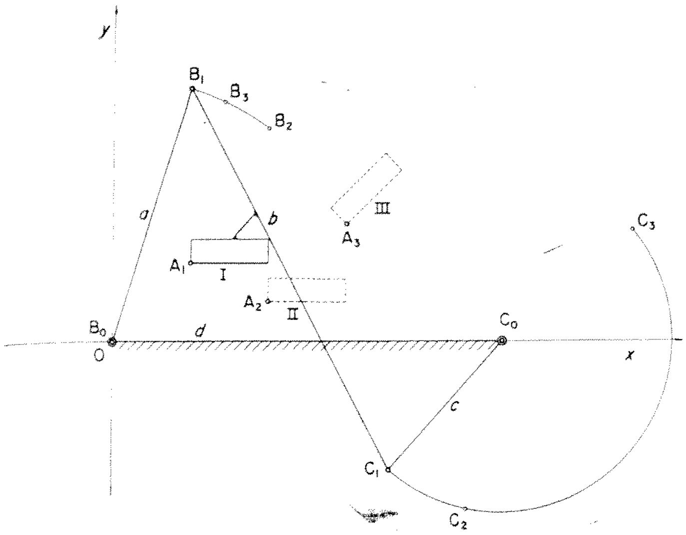

Fig. 4 Example Problem 1: Three-position guidance. Solution with arbitrary choice of two fixed pivots.

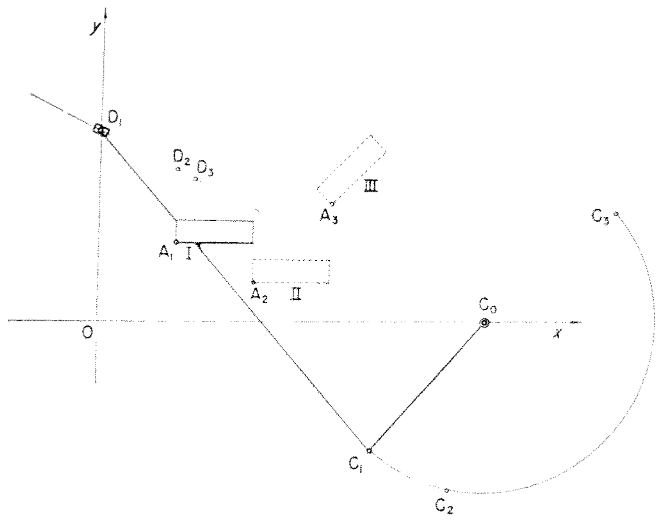

Fig. 5 Example Problems 2: Three-position guidance. Solution with one arbitrary fixed pivot and initial slider position on y-axis.

Numerical solution is by desk calculator to six decimal point accuracy.

Substituting into equation (18) $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (2, 0.5)$, $(X_0, Y_0) = (0, 0)$, and $\theta_{12} = 0$, we have

$$X_1[1 + (-0.5)(0)] + Y_1[(-0.5) - 0] = -{}^1/_2[1 + (-0.5)^2]$$

that is,

$$X_1 - 0.5Y_1 = -0.625000 \qquad (21)$$

Similar substitutions for the displacement to position 3 lead to

$$2.181981X_1 - 2.060661Y_1 = -4.503680 \qquad (22)$$

Solving (21) and (22) simultaneously we obtain

$$X_1 = 0.994078$$

$$Y_1 = 3.238155 \qquad \text{i.e.,} \quad B_1 = B_1(0.994078,\ 3.238155)$$

$$B_0 = B_0(0, 0)$$

Similar substitutions for an assumed pivot $C_0(5, 0)$ lead to a pair of equations in X_1' and Y_1'.

$$X_1' - 0.5Y_1' = 4.375000 \qquad (23)$$

$$3.646446X_1' + 1.474874Y_1' = 10.496320 \qquad (24)$$

which give the coordinates of the second moving pivot in its first position:

$$C_1 = C_1(3.547725, \quad -1.654550)$$

$$C_0 = C_0(5, 0)$$

The completed mechanism is shown in Fig. 4.

Example Problem 2

The conditions of Example Problem 1 are repeated. In this case, however, the guidance is to be accomplished by a slider-crank mechanism such that the fixed crank pivot is located at $C_0 = C_0(5, 0)$ and the slider is located somewhere on the y-axis $(x = 0)$ in its first position.

The choice of fixed crank pivot at $C_0 = C_0(5, 0)$ has already been shown to result in a moving pivot $C_1 = C_1(3.547725, -1.654550)$ and the same crank C_0C_1 will be used in this design. Since for the slider, $X_1 = 0$, we need only to determine the value of Y_1.

From equation (11),

$$[D_{12}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$[D_{13}] = \begin{bmatrix} 0.707107 & -0.707107 & 3 \\ 0.707107 & 0.707107 & 0.085786 \\ 0 & 0 & 1 \end{bmatrix} \qquad (25)$$

we have

$$\begin{bmatrix} X_2 \\ Y_2 \\ 1 \end{bmatrix} = [D_{12}] \begin{bmatrix} 0 \\ Y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ Y_1 - 0.5 \\ 1 \end{bmatrix} \qquad (26)$$

$$\begin{bmatrix} X_3 \\ Y_3 \\ 1 \end{bmatrix} = [D_{13}] \begin{bmatrix} 0 \\ Y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.707107Y_1 + 3 \\ 0.707107Y_1 + 0.085786 \\ 1 \end{bmatrix} \qquad (27)$$

From

$$\begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix} = 0$$

we have $Y_1 = 2.453100$. That is, the slider should be located so that in its first position the slider pivot has the coordinates $D_1 = D_1(0, 2.453100)$. The slope of the slider path is

$$m = \tan\theta = \frac{Y_2 - Y_1}{X_2 - X_1} = \frac{Y_3 - Y_2}{X_3 - X_2} = -0.5$$

$$\theta = \tan^{-1}(-0.5) \doteq -26°35'$$

The completed mechanism is shown in Fig. 5.

7 The Four-Position Guidance Problem

Example Problem 3

A four-position guidance problem has been created, as shown in Fig. 7, by adding a fourth position to the three-position guidance problem of Example Problem 1. Here again, the fact that the first two positions of the member are parallel would cause difficulty in the use of Burmester theory for the graphical determination of the center-point and circle-point curves. In this problem, the four positions of the plane are again specified in terms of the displacement of one point (A) in the plane and the associated angular rotation of the plane.

$$A_1 = A_1(x_1, y_1) = (1, 1)$$

$$A_2 = A_2(x_2, y_2) = (2, 0.5) \qquad \theta_{12} = 0$$

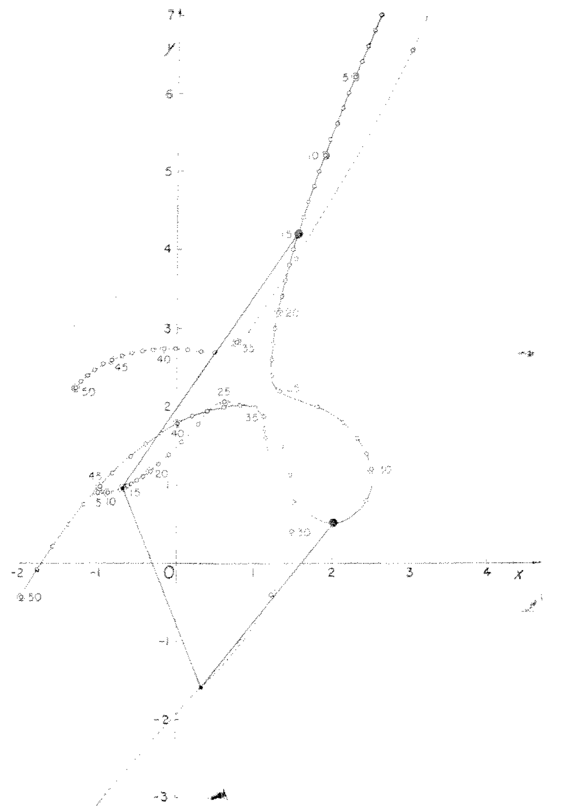

Fig. 6 Example Problem 3: Four-position guidance. Center-point (solid line) and circle-point curves from computer solution. The solution selected is also shown in Fig. 7.

$$A_3 = A_3(x_3, y_3) = (3, 1.5) \qquad \theta_{13} = 45 \text{ deg}$$

$$A_4 = A_4(x_4, y_4) = (2, 2) \qquad \theta_{14} = 90 \text{ deg}$$

$[D_{12}]$ and $[D_{13}]$ have been specified previously in Example Problem 2.

$$[D_{14}] = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Again, letting (X_1, Y_1) be the first position of a moving pivot and (X_0, Y_0) its associated fixed pivot, we may write three matrix equations for the displacements of the moving pivot (X, Y):

$$\begin{bmatrix} X_2 \\ Y_2 \\ 1 \end{bmatrix} = [D_{12}] \begin{bmatrix} X_1 \\ Y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} X_1 + 1 \\ Y_1 - 0.5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_3 \\ Y_3 \\ 1 \end{bmatrix} = [D_{13}] \begin{bmatrix} X_1 \\ Y_1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.707107X_1 - 0.707107Y_1 + 3 \\ 0.707107X_1 + 0.707107Y_1 + 0.085786 \\ 1 \end{bmatrix} \quad (28)$$

$$\begin{bmatrix} X_4 \\ Y_4 \\ 1 \end{bmatrix} = [D_{14}] \begin{bmatrix} X_1 \\ Y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} -Y_1 + 3 \\ X_1 + 1 \\ 1 \end{bmatrix}$$

To insure constant length of the guiding link we have three circle equations,

$$(X_1 - X_0)^2 + (Y_1 - Y_0)^2 = (X_2 - X_0)^2 + (Y_2 - Y_0)^2$$
$$= (X_3 - X_0)^2 + (Y_3 - Y_0)^2 \quad (29)$$
$$= (X_4 - X_0)^2 + (Y_4 - Y_0)^2$$

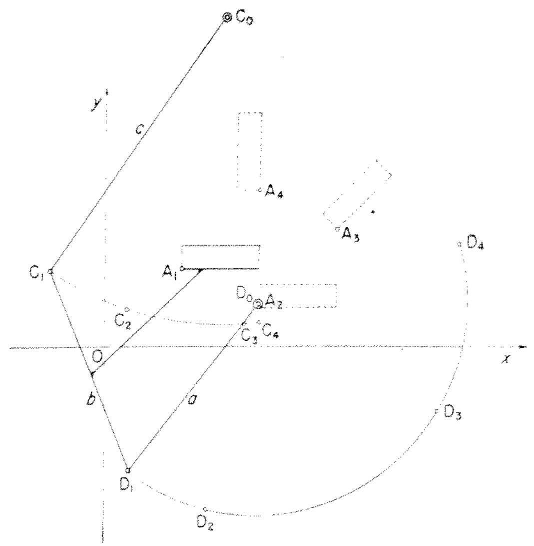

Fig. 7 Example Problem 3: A selected four-bar linkage for four-position guidance

$(X_2, Y_2)(X_3, Y_3)(X_4, Y_4)$ may be specified in terms of X_1, Y_1 as equations (28). Equations (29) may then be considered as a set of three equations in four unknowns. Any one of the four unknowns (X_0, Y_0, X_1, Y_1) may be specified arbitrarily and the set of equations (29) solved for the remaining three.

Although it would be possible to carry out the simultaneous solution of equations (28) and (29) with use of a desk calculator, it was considered more practical to use a digital computer for this purpose.

Equations (28) and (29) constitute a set of nonlinear, second-order simultaneous equations. Such equations are solved easily by existing computer programs. Therefore, a computer program has been prepared which makes use of \$C4 BC SIM4 (R. M. Baer, December, 1961), a computer program available at the University of California Computer Center in either FORTRAN II or FORTRAN IV language.

The four-position plane mechanism synthesis program accepts input data in the form of four positions of a point and three angular displacements. Coordinates of the center-point and circle-point curves are calculated to six decimal accuracy. A plotting subroutine for use with the Cal-Comp plotter may be used to obtain a graphical printout of the center-point and circle-point curves.

Various combinations of guiding links may then be tested either graphically or with an additional computer program which will plot coupler-point curves in a form similar to those shown in Hrones and Nelson [6]. Desired conditions on size, pivot locations, transmission angle, and so on, all linkage dimensions, input-crank angles, and coupler-point path curve coordinates may be checked from the computer printout. The scan region for plotting center-point and circle-point curves may be varied to allow a careful examination of a particular region once a rough decision on pivot location has been made.

Fig. 6 shows the computer printout of the center-point and circle-point curves for the data of Example Problem 3. A particular linkage has been chosen as indicated in Fig. 7.

Example Problem 4

Consider the possibility of a unique slider-crank mechanism which would guide the plane through the sequence of four positions as specified in Example Problem 3. Since four positions are the maximum number of precision positions possible with slider-crank guidance, a solution may or may not exist. Let $D_n(X_n, Y_n)$ $n = 1, 2, 3, 4$ be the coordinates of the slider corresponding to the specified positions of the coupler-point A. The coordinates of D in positions 2, 3, and 4 may be expressed in terms of the coordinates in position 1 with use of the displacement matrix.

The colinearity of points D_1, D_2, D_3, and D_4 is specified by the two straight-line equations

$$\begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_4 & Y_4 & 1 \end{vmatrix} = 0 \qquad (30)$$

Expanding the determinants, we have

$$\frac{Y_2 - Y_1}{X_2 - X_1} = \frac{Y_3 - Y_1}{X_3 - X_1} = \frac{Y_4 - Y_1}{X_4 - X_1} \qquad (31)$$

The solution of equations (31) is carried out as part of the computer program for four-position plane mechanism synthesis. The computer program gave the following results:

$$D_1 = D_1(-1.472791, \quad 1.175736)$$

$$\theta = \tan^{-1}\frac{Y_2 - Y_1}{X_2 - X_1} = \tan^{-1}(-0.5) \doteqdot -26°35'$$

The slope is obvious in this particular problem since the displacement from position 1 to position 2 is a pure translation.

The slider-crank mechanism is shown in Fig. 8. The crank would provide a satisfactory input to the mechanism since the crank rotation is progressive through the positions and always in a counterclockwise direction.

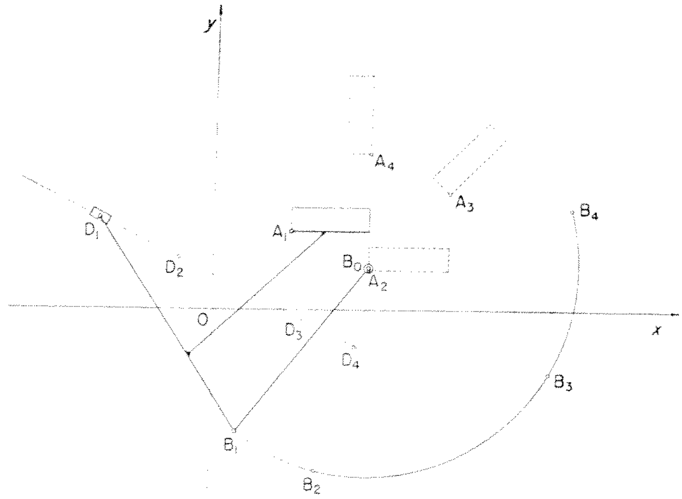

Fig. 8 Example Problem 4: Four-position guidance by slider-crank mechanism

8 Design of Four-Bar Function Generators

Consider the problem of designing a four-bar linkage such that the input-output crank motions are proportional to a specified functional relationship between two variables at a given number of precision points.

The displacement matrix for the relative motion of the input crank with respect to the output crank may be developed by separation of the total relative motion into its components. The input crank has its fixed center $A_0 = A_0(0, 0)$ located at the origin. The fixed center for the output crank is located at $B_0 = B_0(1, 0)$. The total relative motion is composed of a rotation $+\theta_n$ of the input crank followed by a rotation $-\phi_n$ about the output crank center. The $-\phi_n$ component results in an inversion of the entire mechanism about the first position of the output crank.

In those cases where both cranks rotate in the same direction,

$$D_{1n}]_{+R} = [D]_{-\phi_n}[D]_{\theta_n}$$

$$= \left[\begin{array}{cc|c} \cos\phi_n & \sin\phi_n & 1 - \cos\phi_n \\ -\sin\phi_n & \cos\phi_n & \sin\phi_n \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|c} \cos\theta_n & -\sin\theta_n & 0 \\ \sin\theta_n & \cos\theta_n & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$D_{1n}]_{+R} = \left[\begin{array}{cc|c} \cos(\theta_n - \phi_n) & -\sin(\theta_n - \phi_n) & 1 - \cos\phi_n \\ \sin(\theta_n - \phi_n) & \cos(\theta_n - \phi_n) & \sin\phi_n \\ \hline 0 & 0 & 1 \end{array}\right] \qquad (32)$$

hese equations correspond to equation (11) with $\theta = (\theta_n - \phi_n)$, $_1, y_1) = (0, 0)$, and $(x_2, y_2) = (1 - \cos\phi_n, \sin\phi_n)$.

When it is desired to have θ and ϕ with opposite directions of tation, i.e., in a crossed linkage, ϕ_n is replaced by $-\phi_n$ in equa-2), with the result

$$D_{1n}]_{-R} = \left[\begin{array}{cc|c} \cos(\theta_n + \phi_n) & -\sin(\theta_n + \phi_n) & 1 - \cos\phi_n \\ \sin(\theta_n + \phi_n) & \cos(\theta_n + \phi_n) & -\sin\phi_n \\ \hline 0 & 0 & 1 \end{array}\right] \qquad (33)$$

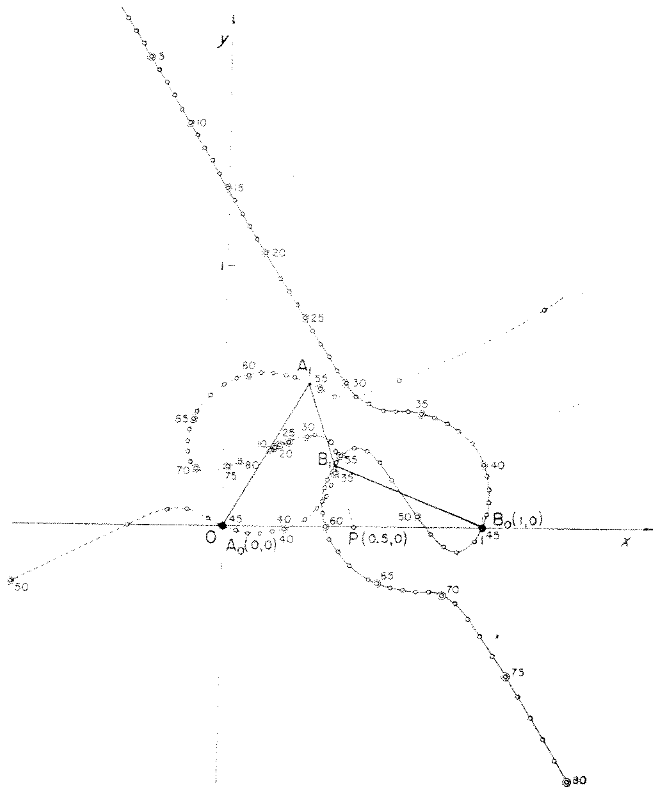

Fig. 9 Example Problem 5: Relative center-point (solid line) and circle-point curves for four-bar linkage to generate $y = e^x$, $0 \leq x \leq 1.2$ and the selected linkage with $A_1(0.316397, 0.553513)$ and $B_1(0.422429, 0.233854)$

Example Problem 5

Design of a four-bar function generator with four precision positions and velocity ratio specified in the first of the four positions.

The design will be arbitrarily assumed to be of the crossed-linkage type.

Function: $y = e^x$
Range: $0 \leq x \leq 1.2$
Input angle: $0 \leq \theta \leq -90$ deg (clockwise)
Output angle: $0 \leq \phi \leq +90$ deg
Precision points: $x = 0, 0.4, 0.8, 1.2$
Velocity ratio: $VR = -1$ at $x = 0$ corresponding to velocity pole at (0.5, 0) when linkage is in first position.

The results of the computer calculations are plotted in Fig. 9 in the form of the loci of the possible moving pivot A_1 and the associated moving pivot B_1, both shown in the first position. The velocity ratio ($VR = -1$) condition is imposed by testing possible couplers A_1B_1 until one is located where the line A_1B_1 passes through the line of centers at point (0.5, 0).

The velocity ratio condition could be added to the computer program by writing the equation of the straight line A_1B_1 which must also include the velocity pole. Note that in the computer program point B_1 would be considered a fixed pivot with coordinates X_0, Y_0.

$$\begin{vmatrix} X_1 & Y_1 & 1 \\ X_P & Y_P & 1 \\ X_0 & Y_0 & 1 \end{vmatrix} = 0 \quad \text{where} \quad \begin{aligned} A_1 &= A_1(X_1, Y_1) \\ B_1 &= B_1(X_0, Y_0) \\ \text{and} \quad \vec{P} &= P(X_P, Y_P) = P(0.5, 0) \end{aligned}$$

In a similar manner, additional velocity ratio conditions may be imposed if at the same time the number of precision positions is reduced.

9 Design of Four-Bar Path Generators

The design of a four-bar mechanism for the guidance of a point $P_1(a_1, b_1)$ through a series of points $P_n(a_n, b_n)$ on a given path is accomplished by the iterative solution of a set of simultaneous nonlinear equations involving the unknown parameters p_0, q_0, p_1, q_1, r_0, s_0, r_1, s_1, and θ_{1n}, as shown in Fig. 10. Since the rotation angle θ_{1n} is listed as an unknown, a new variable will be added for each additional precision point P_n specified along the path. This results in the advantage of either increasing the number of specified path precision points or, alternatively, making possible greater freedom in the arbitrary specification of design parameters. By comparison, in the design of rigid body guidance mechanisms, the coupler rotation angles θ_{1n} are specified as design conditions.

For each position n there are two design equations based upon the constraints imposed by constant length for both links L_2 and L_4.

$$\begin{aligned} (L_2)^2 &= (p_n - p_0)^2 + (q_n - q_0)^2 = (p_1 - p_0)^2 + (q_1 - q_0)^2 \\ (L_4)^2 &= (r_n - r_0)^2 + (s_n - s_0)^2 = (r_1 - r_0)^2 + (s_1 - s_0)^2 \end{aligned} \tag{34}$$

Since

$$\begin{bmatrix} p_n \\ q_n \\ 1 \end{bmatrix} = [D_{1n}] \begin{bmatrix} p_1 \\ q_1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} r_n \\ s_n \\ 1 \end{bmatrix} = [D_{1n}] \begin{bmatrix} r_1 \\ s_1 \\ 1 \end{bmatrix}$$

where $[D_{1n}]$ is the form of equation (11) with $x_1 = a_1$, $x_2 = a_n$, $y_1 = b_1$, and $y_2 = b_n$, it can be shown that equations (34) lead to a pair of design equations suitable for computer programing of the form,

$$\begin{aligned} p_1p_0 + q_1q_0 - a_np_0 - b_nq_0 - a_1p_1 - b_1q_1 + \frac{a_1^2 + b_1^2 + a_n^2 + b_n^2}{2} \\ + \cos\theta_{1n}[-p_0p_1 - q_0q_1 + a_1p_0 + b_1q_0 \\ - a_1a_n - b_1b_n + a_np_1 + b_nq_1] \\ + \sin\theta_{1n}[p_0q_1 - q_0p_1 - b_1p_0 + a_1q_0 \\ + a_nb_1 - a_1b_n + b_np_1 - a_nq_1] = 0 \end{aligned}$$

[Plus a similar expression from the second of equations (34)] (35)

Table 1 summarizes the relationships between number of path precision points, unknown variables, and typical examples of specified variables. It should be noted that, where desired, other combinations may be specified. For example, assume it is desired to use two guiding links of given length L_2 and L_4 in the mechanism; the problem is to locate their fixed and moving pivots in

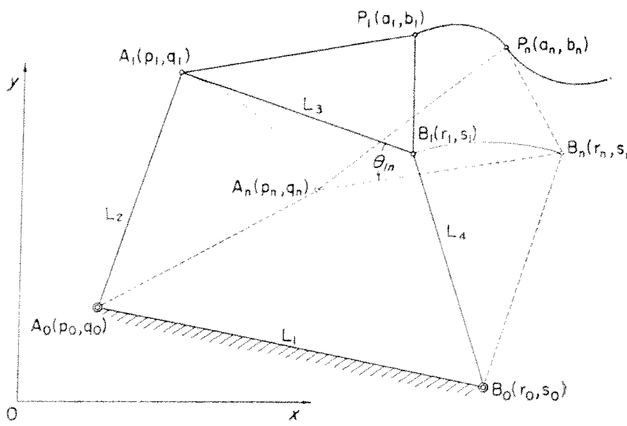

Fig. 10 Four-bar path generator

Table 1 Parametric relationships—four-bar path generator

No. of Precision Points, n	No. of Design Equations	No. of Unknown Variables	No. of Specified Variables		Selected Examples: Synthesis *
2	2	9	7	Specify: Calculate:	$p_0, q_0, p_1, q_1, r_0, s_0, \theta_{12}$ r_1, s_1
3	4	10	6	Specify: Calculate:	$p_0, q_0, p_1, q_1, r_0, s_0$ $r_1, s_1, \theta_{12}, \theta_{13}$
4	6	11	5	Specify: Calculate:	p_0, q_0, p_1, q_1, r_0 $s_0, r_1, s_1, \theta_{12}, \theta_{13}, \theta_{14}$
5	8	12	4	Specify: Calculate:	p_0, q_0, r_0, s_0, $p_1, q_1, r_1, s_1, \theta_{12}, \theta_{13}, \theta_{14}, \theta_{15}$ (See Example Problem 6)
6 †	10	13	3	Specify: Calculate:	p_0, q_0, r_0 all others
7 †	12	14	2	Specify: Calculate:	p_0, r_0 all others
8 †	14	15	1	Specify: Calculate:	p_0 all others
9 †	16	16	0	Possible unique solution	

*Symbols represent coordinates as shown in Figure 10.

†Solutions highly dependent upon initial guesses and numerical accuracy.

the first position. This specification would add the two following equations to equations (35).

$$\begin{aligned} (L_2)^2 &= (p_1 - p_0)^2 + (q_1 - q_0)^2 \\ (L_4)^2 &= (r_1 - r_0)^2 + (s_1 - s_0)^2 \end{aligned} \tag{36}$$

In the case of five-precision point path generation, there would be 10 equations and 12 unknowns, and two parameters may be specified, e.g., p_0, q_0.

Example Problem 6

Design of a path generation mechanism with five path precision points.

Fig. 11 illustrates a four-bar mechanism for the guidance of a point P through precision points.

$$\begin{aligned} P_1(a_1, b_1) &= (1.00000, 1.00000) \\ P_2(a_2, b_2) &= (2.00000, 0.50000) \\ P_3(a_3, b_3) &= (3.00000, 1.50000) \\ P_4(a_4, b_4) &= (2.00000, 2.00000) \\ P_5(a_5, b_5) &= (1.50000, 1.90000) \end{aligned}$$

In this example, fixed pivots were initially assumed at

$$\begin{aligned} A_0(p_0, q_0) &= (2.10000, 0.60000) \\ B_0(r_0, s_0) &= (1.50000, 4.20000) \end{aligned}$$

The computer solution gave

$$A_1(p_1, q_1) = (0.6073749, \quad -1.127103)$$

$$B_1(r_1, s_1) = (-0.5863996, 0.9969990)$$

The computer program will also determine the loci of possible moving pivots A_1 and B_1 for arbitrary finite changes in the location of either fixed pivot. These loci are shown in Fig. 11 for 13 positions of $B_0(r_0, s_0)$ spaced uniformly on a straight line between $B_0(1.50000, 4.20000)$ and $B_0(-1.50000, 3.00000)$.

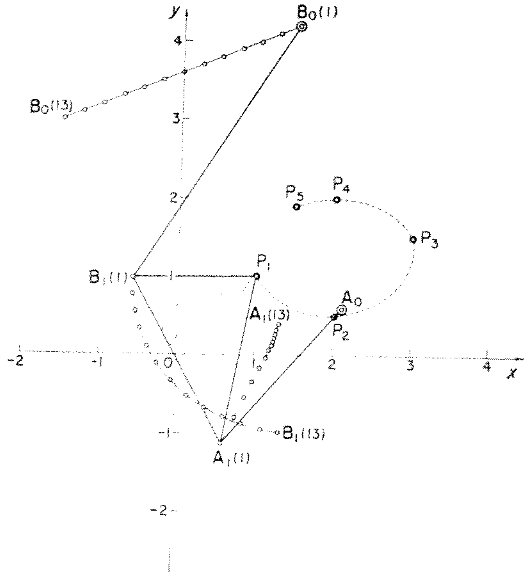

Fig. 11 Example Problem 6: Four-bar linkage for path generation with five precision points and arbitrary choice of both fixed pivots. One fixed pivot varied to display loci of two moving pivots.

Example Problem 7

Design of a path generation mechanism with the same five path precision points of Example Problem 6 but with fixed pivot specified only at

$$A_0(p_0, q_0) = (2.10000, 0.50000)$$

Two cranks of arbitrary length are specified as $L_2 = 1.0$ unit and $L_4 = 2.0$ units. The computer solution gave

$$B_0(r_0, s_0) = (0.6934239, 1.184073)$$

$$A_1(p_1, q_1) = (1.206753, 0.05043468)$$

$$B_1(r_1, s_1) = (0.3341094, -0.7833851)$$

The computer program will also determine the loci of possible moving pivots A_1 and B_1 and of one fixed pivot B_0 for arbitrary finite changes in the location of the assumed fixed pivot A_0. These loci are shown in Fig. 12 for seven positions of $A_0(p_0, q_0)$ spaced uniformly on a vertical straight line between $A_0(2.10000, 0.50000)$ and $A_0(2.10000, 1.10000)$.

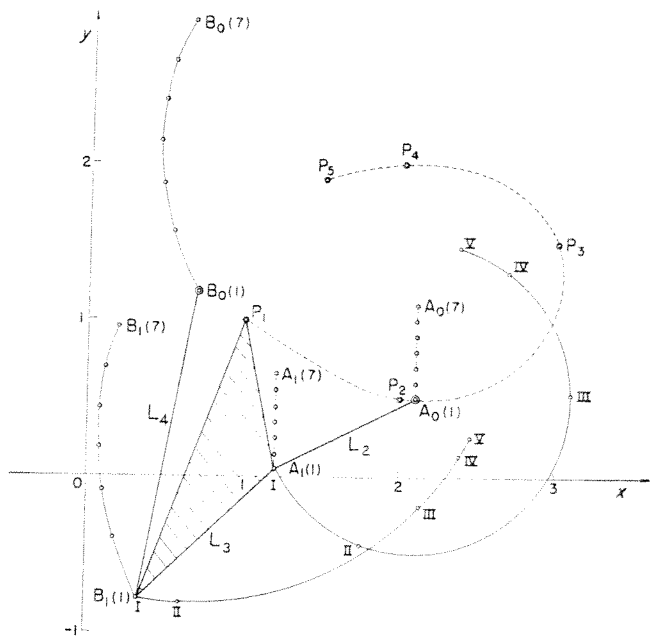

Fig. 12 Example Problem 7: Four-bar linkage for path generation with five precision points. Arbitrary specification of one fixed pivot and lengths of both guiding cranks. One fixed pivot varied to display loci of two moving pivots and second fixed pivot.

10 Design of Geared Five-Bar Function Generators

Sandor [2] has discussed the use of geared five-bar linkages as function generators and has shown the possibility of a six-precision-point solution with specified gear ratio obtained by the complex-number method.

With use of the displacement matrix methods of the present paper, it is possible to develop a seven-precision-point solution for an arbitrarily assumed gear ratio R. A maximum of eight precision points is theoretically possible if R is considered as a variable in the equations. The practical realization of such a solution may be difficult when gears with finite number of teeth are used. Use of crossed belts or friction wheels would be a possibility.

The design equation is developed by considering the motion of link 4 relative to link 2, as shown in Fig. 13. The displacement matrix for link 4 is formed by considering successive rotations about points B_1, B_0, and A_0; this procedure leads to equation (37). The input angle is θ_{1n} and the output angle is ϕ_{1n}.

$$[D_{1n}]_{4/2} = \left[\begin{array}{cc|c} \cos\beta_{1n} & -\sin\beta_{1n} & r_1(\cos\alpha_{1n} - \cos\beta_{1n}) - s_1(\sin\alpha_{1n} - \sin\beta_{1n}) + (1 - \cos\phi_{1n}) \\ \sin\beta_{1n} & \cos\beta_{1n} & r_1(\sin\alpha_{1n} - \sin\beta_{1n}) + s_1(\cos\alpha_{1n} - \cos\beta_{1n}) + \sin\phi_{1n} \\ \hline 0 & 0 & 1 \end{array}\right]$$

where

$$\alpha_{1n} = (\theta_{1n} - \phi_{1n})$$
$$\beta_{1n} = (\theta_{1n} - \phi_{1n} + R\theta_{1n}) \qquad (37)$$

From the condition L_3 = constant there will be $(n - 1)$ design equations which may be expressed in a form suitable for programing as,

$$(u_1 - p_1)^2 + (v_1 - q_1)^2$$
$$= [C_{1n}u_1 - D_{1n}v_1 + r_1(A_{1n} - C_{1n}) - s_1(B_{1n} - D_{1n}) + E_{1n} - p_1]^2$$
$$+ [D_{1n}u_1 + C_{1n}v_1 + r_1(B_{1n} - D_{1n}) + s_1(A_{1n} - C_{1n}) + F_{1n} - q_1]^2$$

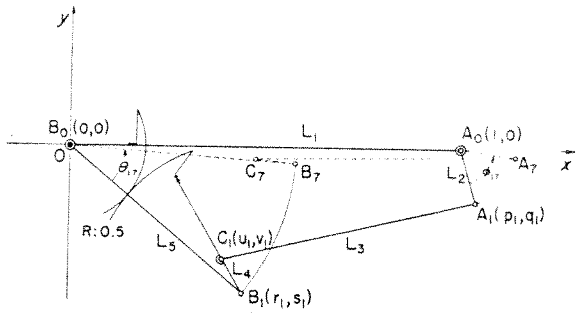

Fig. 13 Example Problem 8: Geared five-bar function generator with seven precision points. Gear ratio arbitrarily specified as +0.50.

where

$$A_{1n} = \cos \alpha_{1n}$$

$$B_{1n} = \sin \alpha_{1n}$$

$$C_{1n} = \cos \beta_{1n}$$

$$D_{1n} = \sin \beta_{1n}$$

$$E_{1n} = 1 - \cos \phi_{1n}$$

$$F_{1n} = \sin \phi_{1n} \qquad (38)$$

Table 2 gives an analysis of the parametric relationships for geared five-bar function generators and indicates the possiblity of a maximum of eight precision points for the geared five-bar linkage. A seven-point solution is presented as an example.

Example Problem 8

Design of a geared five-bar function generator with seven precision points with arbitrary gear ratio $R = 0.5$.

Input and output angles were assumed according to the following schedule.

$$\theta_{12} = 10.0 \text{ deg} \qquad \phi_{12} = 20.0 \text{ deg}$$

$$\theta_{13} = 18.0 \text{ deg} \qquad \phi_{13} = 34.5 \text{ deg}$$

$$\theta_{14} = 20.0 \text{ deg} \qquad \phi_{14} = 38.0 \text{ deg}$$

$$\theta_{15} = 30.0 \text{ deg} \qquad \phi_{15} = 55.0 \text{ deg}$$

$$\theta_{16} = 33.0 \text{ deg} \qquad \phi_{16} = 60.0 \text{ deg}$$

$$\theta_{17} = 36.0 \text{ deg} \qquad \phi_{17} = 65.0 \text{ deg}$$

The parameters of the resulting mechanism, found by the computer program, are

$$p_1 = 1.042552$$

$$q_1 = -0.1362623$$

$$u_1 = 0.3900990$$

$$v_1 = -0.2806455$$

$$r_1 = 0.4497287$$

$$s_1 = -0.3728327 \quad \text{with base link} \quad L_1 = \text{unity}$$

The solution is shown in Fig. 13 with $R = 0.5$. It should be noted that there are no restrictions on the choice of gear ratio R except as dictated by the practical choice of number of teeth on mating gears. A number of gear ratios could be investigated simultaneously by plotting loci of moving pivot locations as a function of assumed gear ratio.

Using the methods of the present paper, the authors have also obtained solutions for path generation mechanisms of the two-gear five-bar and the four-gear six-bar types. In these cases also, solutions may be obtained for an arbitrary choice of gear ratio with corresponding reduction in number of path precision points.

Table 2 Parametric relationships—geared five-bar function generator

No. of Precision Points, n	No. of Design Equations	No. of Unknown Variables	No. of Specified Variables		Selected Examples: Synthesis *
2	1	7	6	Specify: Calculate:	$R, r_1, s_1, u_1, v_1, p_1$ q_1
3	2	7	5	Specify: Calculate:	R, r_1, s_1, u_1, v_1 p_1, q_1
4	3	7	4	Specify: Calculate:	R, r_1, s_1, u_1 v_1, p_1, q_1
5	4	7	3	Specify: Calculate:	R, r_1, s_1 u_1, v_1, p_1, q_1
6	5	7	2	Specify: Calculate:	R, r_1 s_1, u_1, v_1, p_1, q_1
7	6	7	1	Specify: Calculate: [See Example Problem 8]	R $r_1, s_1, u_1, v_1, p_1, q_1$
8	7	7	0	Possible unique solution for a calculated value of R	

*Symbols represent coordinates as shown in Figure 13.

Tables 1 and 2 indicate the possibility of an increase in number of precision points in certain design situations as compared with existing complex number methods. As noted, in some cases the solutions obtained are highly dependent upon the initial guesses in the numerical iteration scheme. Work is continuing on approximate methods of solution leading to an improved set of initial values such that final convergence to an accurate solution can be assured in problems involving larger numbers of unspecified parameters.

Computer programs written in FORTRAN IV language for the synthesis of rigid body guidance, function generation and path generation mechanisms are available from the authors upon request.

11 Conclusion

The numerical methods based upon the displacement matrix have proved to be useful in the synthesis of any type of plane mechanism constrained by lower pairs. The method is suitable for desk calculator computation with up to three precision points and makes efficient use of digital computers for larger numbers of precision points.

Acknowledgments

The present work is a part of a study of mathematical methods useful in the design of three-dimensional exoskeletal systems for the human body, particularly orthopaedic braces for the lower extremity. The support of the Veterans Administration, through Contract No. V1005M-2075 with the Prosthetic and Sensory Aids Service, is gratefully acknowledged.

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