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The Generalized Principle of Inertia Match for Geared Robotic Mechanisms

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Abstract

The principle of inertia match has been extended from one-degree-of-freedom(D.O.F.) system to multi-D.O.F. systems. Based on the concept of maximum acceleration capacity, a methodology for the determination of gear ratios in geared robotic mechanisms has been developed. It is found that, at the optimum design, the mass inertia matrix of the input links reflected at the joint-space is equal to that of the major links, and the maximum acceleration capacity is independent of the gear train arrangement. Several two-D.O.F. geared robotic mechanisms have been used as design examples to illustrated the principle. Using this methodology, mechanisms can be designed to yield optimum dynamic performance.

I. Introduction

Various performance measures such as the velocity ellipsoid and the generalized velocity ratio [1, 6], the condition number [7], and the dynamic manipulability index [10] have been proposed for the evaluation of manipulators. Since these performance measures are based on the transformation between the "joint-space" and the "end-effector-space", they are useful for the evaluation and/or design of direct-drive manipulators. However, they are not very helpful for the evaluation and design of manipulators using gear trains or other means for power transmission. For geared robotic mechanisms, the transformation between the "joint-space" and the "actuator-space" must also be considered. Taking this into consideration, Chen and Tsai [4] defined the generalized velocity ratio and acceleration capacity for the design and performance evaluation of geared robotic mechanisms.

For one-D.O.F. geared mechanisms, the principle of inertia match [8] can be used as a guideline for the selection of gear ratios. As for multi-D.O.F. mechanisms, an approach based on kinematic isotropy followed by acceleration capacity optimization was proposed and the concept of two-stage gear-reduction was introduced for the determination of gear ratios by Chen and Tsai [4]. In this paper, a new approach based on the optimization of acceleration capacity alone will be presented. Design equations and optimality conditions will be derived. Several two-D.O.F. robotic mechanisms will be used as numerical examples to demonstrate the principle.

II. Kinematic Equations

In this section, some kinematic equations for geared robotic mechanisms will be briefly reviewed. Figure 1 shows a geared robotic mechanism in conceptual form, where the inputs to the mechanism are the actuators and the output is the end-effector. Let Φ, Θ, and X be the displacement vectors associated with the actuators, joints, and the end-effector. Then, the joint velocity vector, $\dot{\Theta}$, and the output velocity vector, $\dot{X}$, are related by the Jacobian matrix, J, as

$$\dot{X} = J\,\dot{\Theta} \tag{1}$$

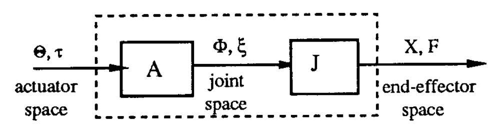

Fig. 1: Conceptual diagram of a geared robotic mechanism

And the actuator velocity vector, $\dot{\Phi}$, is related to the joint velocity vector, $\dot{\Theta}$, by

$$\dot{\Phi} = A^T\,\dot{\Theta} \tag{2}$$

where $(\,)^T$ denotes the transpose of ( ). We note that A is the structure matrix whose elements are functions of gear ratios and each column of A represents a transmission line in a mechanism[3].

Similarly, the joint torque, τ, is related to the external force vector F by

$$\tau = J^T\,F \tag{3}$$

The joint torque, τ, is related to the actuator torque, ξ, by

$$\tau = A\,\xi \tag{4}$$

III. Dynamic Equations

A. Principle of Inertia Match

Figure 2a shows a one-D.O.F. geared mechanism. The equation of motion is

$$(I_r + I_L)\ddot{q} = g\,\xi_i \tag{5}$$

where $I_r = I_i\,g^2$ denotes the inertia of the input link reflected at the output shaft, I_i the inertia of the input link, I_L the inertia of the output link, ξ_i the input torque, q the angular displacement of the output shaft, and $g = N_2/N_1$ the gear ratio.

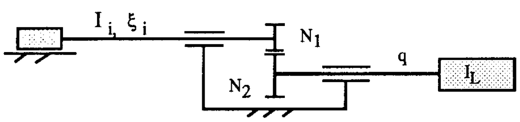

Fig. 2(a): A one-D.O.F. geared mechanism.

Assume that I_i and I_L remain constant regardless of the change in gear ratio and assume that there is no power loss in the gear mesh. Fig. 2b shows the relation between the output shaft acceleration, $\ddot{q}$, and the gear ratio, g. It is clear that, given ξ_i, I_i and I_L, there exists an optimum gear ratio which yields a maximum output acceleration. At the optimum design, the output acceleration and the gear ratio are given by

$$\ddot{q})_{max} = \frac{\xi_i}{2\sqrt{I_i I_L}} \tag{6}$$

$$g^2_{opt} = \frac{I_L}{I_i} \tag{7}$$

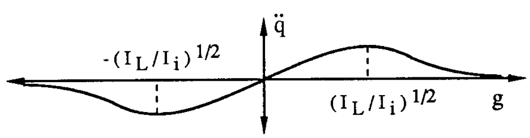

Fig. 2(b): Output acceleration vs. gear ratio.

It can be said that the gear ratio is chosen such that the reflected input inertia is "matched" with the output inertia. This is known as "principle of inertia match" [8].

B. Acceleration Capacity

The equations of motion for an n-D.O.F. geared robotic mechanism can be written in the joint-space as [5]

$$M\,\ddot{\Theta} + \dot{\Theta}^T C\,\dot{\Theta} + G = A\,\xi \tag{8}$$

where M is an n by n symmetric inertia matrix, $\dot{\Theta}^T C\,\dot{\Theta}$ is the generalized inertia force contributed by the Coriolis and centrifugal effects, and G is the generalized active force contributed by gravitational effect and external loads.

In what follows, we shall neglect the Coriolis and centripetal forces, and we shall also assume that there are no gravitational forces and external loads. Under this assumption, eq. (8) reduces to

$$M\,\ddot{\Theta} = A\,\xi \tag{9}$$

Differentiating eq. (1) with respect to time and neglecting the Coriolis and centrifugal effects, we obtain

$$\ddot{X} = J\,\ddot{\Theta} \tag{10}$$

Eliminating $\ddot{\Theta}$ from eqs. (9) and (10), yields

$$A^{-1} M\, J^{-1}\, \ddot{X} = \xi \tag{11}$$

Equation (11) provides a torque transformation from the end-effector-space to the actuator-space. Note that both matrices A and M are functions of gear ratios.

The question we want to answer is

$$\text{Given} \quad |\xi|^2 \equiv \xi^T W_\xi\, \xi = 1, \tag{12}$$

what gear ratios yield the optimum dynamic performance? In eq. (12), W_ξ is a diagonal, positive definite, weighting matrix.

Substituting eq. (11) and its transpose into (12), we obtain

$$|\xi|^2 = \ddot{X}^T J^{-T} M^T A^{-T} W_\xi A^{-1} M\, J^{-1}\, \ddot{X} = 1 \tag{13}$$

Equation (13) represents an acceleration ellipsoid in the end-effector space. As an extension of the principle of inertia match, Chen and Tsai [4] defined the acceleration capacity (A.C.) to be proportional to the volume of the acceleration ellipsoid. They showed that

$$\text{A.C.} = \frac{[\det(J^T W_x\, J)\det(A\, W_\phi\, A^T)]^{1/2}}{\det(M)} \tag{14}$$

where W_x and W_ϕ are diagonal, positive definite, weighting matrices.

The problem we want to solve now becomes:

$$\text{Given} \quad |\xi|^2 = 1,$$

what gear ratios yield the optimum acceleration capacity? To answer this question, we will first examine the inertia matrix M, and then seek for the optimum solution.

IV. The Inertia Matrix M

It has been shown that there exists an "equivalent open-loop chain" in a geared robotic mechanism [9]. Each link in the equivalent open-loop chain is referred to as a major link while all the other links are called the carried links [5]. As shown in Fig. 3, links 1, 2 and 3 are the major links, and links 4 and 5 are the carried links.

In order to facilitate the dynamic analysis, Chen [5] suggested the following approach. First, all the carried links are treated as being rigidly attached to their carriers and the generalized inertia forces due to the resultant equivalent open-loop linkage are formulated. Second, the effects of relative rotations of the carried links with respect to their carriers are formulated and added to the generalized inertia forces. Let M_m and M_r be the inertia matrices due to the first and second part of the aforementioned generalized inertia forces, respectively. Then, the inertia matrix M can be written as

$$M = M_m + M_r \tag{15}$$

where both M_m and M_r are positive definite symmetric matrices.

The kinetic energy of carried link i due to relative rotation with respect to its carrier j can be written as[5]

$$K_{i,j} = \frac{1}{2} I_i \dot{\theta}_{i,j}^2 + I_i \dot{\theta}_{i,j} (\omega_j \cdot v_i) \tag{16}$$

where $K_{i,j}$ denotes the kinetic energy of link i due to its rotation with respect to link j, v_i a unit vector along the "positive" axis of rotation of link i, I_i the moment of inertia of link i about its axis of rotation, $\dot{\theta}_{i,j}$ rotational speed of link i with respect to link j, and w_j angular velocity vector of carrier j with respect to the inertia frame.

The angular velocity of a major link j, in an open-loop chain, can be written as

$$\omega_j = \sum_{s=1}^{j-1} (Z_s \dot{q}_s) \tag{17}$$

where Z_s denotes a unit vector along the s-th joint axis in the equivalent open-loop chain, and $\dot{q}_s$ the rate of change of the joint angle q_s. We note that the unit vectors Z_s, s = 1, 2,···, j-1, are functions of the joint angles.

With the fundamental-circuit equations and the appropriate coaxiality conditions [9], the rotational speed of carried link i with respect to its carrier j can be written as a linear summation of the joint rates as shown below:

$$\dot{\theta}_{i,j} = \sum_{s=j}^{n} (b_{is} \dot{q}_s) \tag{18}$$

where b_{is}, s = j, j+1,···, n, are functions of gear ratios. Furthermore, b_{is} are the elements of the r-th column in the structure matrix A defined in [3] if link i is the input link on r-th transmission line, and a collection of these $\dot{\theta}_{i,j}$'s forms the actuator velocity vector $\dot{\Phi}$.

Substituting eqs. (17) and (18) into (16), we obtain

$$K_{i,j} = I_i \{ [\sum_{s=j}^{n} (b_{is}\dot{q}_s)]^2/2 + \sum_{s=j}^{n} (b_{is}\dot{q}_s) \sum_{s=1}^{j-1} (Z_s \dot{q}_s)\cdot v_i \} \tag{19}$$

Applying Lagrangian equation on eq. (19) and neglecting the Coriolis and centrifugal terms, we obtain

$$F^*_{ir} = I_i b_{ir} [\sum_{s=j}^{n} (b_{is}\ddot{q}_s) + \sum_{s=1}^{j-1} (Z_s \ddot{q}_s)\cdot v_i], \quad \text{for } r \geq j \tag{20a}$$

$$F^*_{ir} = I_i [\sum_{s=j}^{n} (b_{is}\ddot{q}_s)(Z_r \cdot v_i)], \quad \text{for } r < j \tag{20b}$$

where F^*_{ir} denotes the generalized inertia force due to the relative motion of a carried link i with respect to its carrier j, and associated with q_r. Note that the order-of-magnitude for ($Z_s \cdot v_i$) ranges from -1 to +1, while the b_{is}'s are usually one order-of-magnitude larger than ($Z_s \cdot v_i$). Hence, in general, the first term in eq. (20a) dominates the equation and eq. (20) can be approximated as:

$$F^*_{ir} = \begin{cases} I_i b_{ir} \sum_{s=j}^{n} (b_{is} \ddot{q}_s), & \text{for } r \geq j \\ 0, & \text{for } r < j \end{cases} \tag{21}$$

Hence, the contribution of input links to the inertia matrix M_r can be obtained by assembling the coefficients of $\ddot{q}_s$ in eq. (21), for all combination of i and r, as

$$M_r = I_m A U A^T \tag{22}$$

where

$$I_m = (\prod_{i=1}^{n} I_i)^{1/n} \tag{23}$$

and I_i is the inertia of i-th input link, U is a diagonal scaling matrix with its (i, i) element equal to I_i / I_m and its determinant equal to unity. Note that the contribution to the inertia matrix M_r due to other carried links have been neglected, since they are usually one order-of-magnitude smaller than that due to the input links. It should also be noted that M_r is a function of gear ratios while M_m is a function of the joint angles and the link mass properties. Hence, we can optimize the design of a manipulator only at a predetermined manipulator posture.

V. Acceleration Capacity Optimization

Taking the determinant of eq. (22), yields

$$\det(M_r) = \det(I_m A U A^T) = I_m^n \det(A A^T) \tag{24}$$

From eq. (24), eq. (14) can be further reduced to

$$\text{A.C.} = \alpha \lambda \tag{25}$$

where

$$\alpha = \left(\frac{\det(J^T W_x J)\det(W_\Phi)}{I_m^n}\right)^{1/2} \tag{26}$$

$$\lambda = \frac{\det(M_r)^{1/2}}{\det(M)} \tag{27}$$

Note that to maximize acceleration capacity is equivalent to maximize λ, since α is a constant at a given posture.

A. Two-D.O.F. Systems:

Assume that the structure matrix A takes the following general form:

$$A = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \tag{28}$$

Then, from eq. (22), inertia matrix M_r can be written as

$$M_r = \begin{bmatrix} \kappa_1 & \kappa_2 \\ \kappa_2 & \kappa_3 \end{bmatrix} \tag{29}$$

where

$$\kappa_1 = I_1 g_{11}^2 + I_2 g_{12}^2 \tag{30}$$

$$\kappa_2 = I_1 g_{11} g_{21} + I_2 g_{12} g_{22} \quad (31)$$

$$\kappa_3 = I_1 g_{21}^2 + I_2 g_{22}^2 \quad (32)$$

Note that the matrix M_r contains only three independent parameters, although the number of non-zero elements in the structure matrix can be as many as four. Also note that the structure matrix must have at least three non-zero elements in order for M_r to be non-singular and to have non-zero k_2.

Similarly, the inertia matrix M_m can also be expressed in terms of three independent parameters as shown below:

$$M_m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \quad (33)$$

Hence, from eq. (15), the inertia matrix M is given by

$$M = \begin{bmatrix} m_1 + \kappa_1 & m_2 + \kappa_2 \\ m_2 + \kappa_2 & m_3 + \kappa_3 \end{bmatrix} \quad (34)$$

Substituting the determinants of eqs. (29) and (34) into eq. (27), we obtain

$$\lambda = \frac{(\kappa_1 \kappa_3 - \kappa_2^2)^{1/2}}{(m_1 + \kappa_1)(m_3 + \kappa_3) - (m_2 + \kappa_2)^2} \quad (35)$$

Taking the derivative of λ with respect to k_i for i = 1, 2 and 3, and equating them to zero, we obtain

$$(m_3+\kappa_3)[\kappa_3(m_1-\kappa_1)+2\kappa_2^2]-\kappa_3(m_2+\kappa_2)^2 = 0 \quad (36a)$$

$$(m_2+\kappa_2)[\kappa_2(m_2-\kappa_2)+2\kappa_1\kappa_3]-\kappa_2(m_1+\kappa_1)(m_3+\kappa_3) = 0 \quad (36b)$$

$$(m_1+\kappa_1)[\kappa_1(m_3-\kappa_3)+2\kappa_2^2]-\kappa_1(m_2+\kappa_2)^2 = 0 \quad (36c)$$

Two non-trivial solutions to eqs. (36a)-(36c) are:

$$\begin{cases} \kappa_1 = m_1 \\ \kappa_2 = m_2 \\ \kappa_3 = m_3 \end{cases} \quad \text{and} \quad \begin{cases} \kappa_1 = -m_1 \\ \kappa_2 = -m_2 \\ \kappa_3 = -m_3 \end{cases} \quad (37)$$

Since the inertias must be non-negative real numbers, only the former set is a feasible solution. In other word, for two-D.O.F. systems, the optimality condition for maximum acceleration capacity is

$$M_r)_{i,j} = M_m)_{i,j} \quad (38)$$

provided M_r and M_m have the same number of independent parameters. Substituting eq. (38) into eq. (15) and the resulting equation into eq. (27), we obtain

$$\lambda)_{opt} = \frac{1}{2^2 \det(M_m)^{1/2}} \quad (39)$$

From eqs. (25) and (39), we note that, given J, I_1 and I_2, the maximum acceleration capacity of a manipulator at a prescribed posture is independent of the gearing configuration, i.e. the arrangement of transmission lines.

B. N-D.O.F. Systems:

In the appendix, we have proved that

$$\frac{\det(M_r)^{1/2}}{\det(M)} \le \frac{1}{2^n \det(M_m)^{1/2}} \quad (40)$$

and

$$M_r)_{i,j} = M_m)_{i,j} \quad (41)$$

is a sufficient condition for the equality sign to hold. This leads to the following theorem.

Theorem: For n-D.O.F. geared robotic systems, the acceleration capacity is bounded by the following inequality:

$$A.C. \le \frac{1}{2^n} \left(\frac{\det(J^T W_x J)\det(W_\phi)}{I_m^n \det(M_m)}\right)^{1/2} \quad (42)$$

A sufficient condition for the sign of equality to hold is

$$M_r = M_m \quad (43)$$

Equation (43) requires the forms of M_r and M_m to be compatible. When the equality sign holds, the optimum value of acceleration capacity at a given posture is independent of the gearing configuration.

Equation (43) implies that, at the optimum design, the mass inertia matrix of the input links reflected at the joint-space is equal to that of the major links. We shall call the above theorem the *generalized principle of inertia match* for multi-D.O.F. geared robotic systems.

VI. Design Examples

For the two-D.O.F. planar manipulators as shown in Figs. 3-4, assume that, at a given posture, the inertia matrix M_m takes the form of eq. (34) and the product of Jacobian matrix is:

$$J^T W_x J = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad (44)$$

The effect of gearing configuration on the optimum gear ratios is discussed as follows:

A. Individual Joint-Drive Manipulator

Figure 3 shows a individual joint-drive manipulator [4]. The structure matrix A can be written as

$$A = \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22} \end{bmatrix} \quad (45)$$

Substituting eq. (45) into eqs. (30)-(32) and the resulting equations into eq. (29), we obtain

$$M_r = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_3 \end{bmatrix} \quad (46)$$

where

$$\kappa_1 = I_1 g_{11}^2 \quad (47a)$$

$$\kappa_3 = I_2\, g_{22}^2 \tag{47b}$$

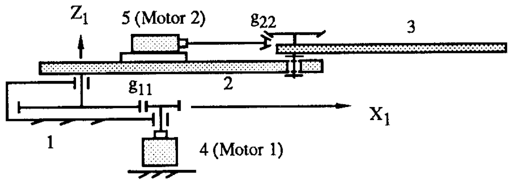

Fig. 3: A two-D.O.F. individual joint-drive manipulator.

Since $k_2 = 0$, the forms of M_r and M_m are not compatible with each other and, therefore, eq. (38) can not be used as a valid solution.

With $k_2 = 0$, eqs. (36a) and (36c) reduce to

$$[(m_1 m_3 - m_2^2) - \kappa_1 \kappa_3] - (\kappa_1 m_3 - \kappa_3 m_1) = 0 \tag{48a}$$

$$[(m_1 m_3 - m_2^2) - \kappa_1 \kappa_3] + (\kappa_1 m_3 - \kappa_3 m_1) = 0 \tag{48b}$$

From eqs. (47a-b) and (48a-b), the optimal conditions for the individual joint-drive manipulator are

$$m_1 m_3 - m_2^2 = I_1 I_2\, g_{11}^2\, g_{22}^2 \qquad (49b)$$

$$I_1\, g_{11}^2\, m_3 = I_2\, g_{22}^2\, m_1 \qquad (49b)$$

Solving eqs. (49a-b) for g_{11} and g_{22} and substituting the results into eq. (45), we have

$$A = \begin{bmatrix} \left(\dfrac{\rho_1 m_1}{I_1^2 m_3}\right)^{1/4} & 0 \\ 0 & \left(\dfrac{\rho_1 m_3}{I_2^2 m_1}\right)^{1/4} \end{bmatrix} \tag{50}$$

where

$$\rho_1 = m_1 m_3 - m_2^2 \tag{51}$$

Assuming W_ϕ is an identity matrix, then from eq. (25), the maximum A.C. can be written as

$$A.C.)_{max} = \frac{(a c - b^2)^{1/2}}{2 (I_1 I_2)^{1/2} (\rho_1^{1/2} + (m_1 m_3)^{1/2})} \tag{52}$$

B. Gear-Coupled Manipulator

Figure 4 shows a gear-coupled manipulator having three non-zero elements in its structure matrix as shown below:

$$A = \begin{bmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{bmatrix} \tag{53}$$

where $g_{22} = g_{12}\, n_{12}\, n_{22}$. Substituting eq. (53) into eq. (39) and solving the resulting equations we obtain

$$A = \begin{bmatrix} \left(\dfrac{\rho_1}{I_1 m_3}\right)^{1/2} & \dfrac{m_2}{(I_2 m_3)^{1/2}} \\ 0 & \left(\dfrac{m_3}{I_2}\right)^{1/2} \end{bmatrix} \tag{54}$$

and from eqs. (25), (26) and (39), we obtain

$$A.C.)_{max} = \frac{1}{4} \left(\frac{a c - b^2}{I_1 I_2 \rho_1}\right)^{1/2} \tag{55}$$

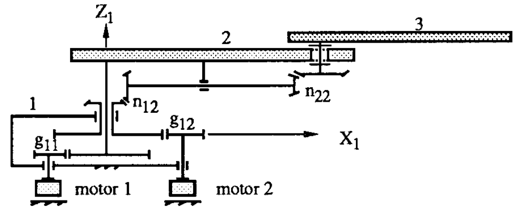

Fig. 4: A two-D.O.F. planar gear-coupled manipulator.

Note that a sign change along any column of the structure matrices as shown in eqs. (50) and (54) does not change the optimum acceleration capacity[4].

VII. Numerical Evaluation

For the two-D.O.F. planar manipulators as shown in Figs. 3-4, it can be shown that the Jacobian matrix is given by

$$J = \begin{bmatrix} -d_3 S_{12} - d_2 S_1 & -d_3 S_{12} \\ d_3 C_{12} + d_2 C_1 & d_3 C_{12} \end{bmatrix} \tag{56}$$

where $d_2 = 22.86$ cm, $d_3 = 17.78$ cm are the lengths of link 2 and link 3, respectively, and where S_i, C_i, S_{12}, and C_{12} denote $\sin(\theta_i)$, $\cos(\theta_i)$, $\sin(\theta_1+\theta_2)$, and $\cos(\theta_1+\theta_2)$, respectively. With the end-effector positioned at $[X_1, Y_1] = [22.86, 0]$ as the design reference point, we have (See [4] for detailed derivation)

$$M_m = \begin{bmatrix} 958 & 29.4 \\ 29.4 & 107 \end{bmatrix} \quad (\text{kg} - \text{cm}^2) \tag{57}$$

and

$$J = \begin{bmatrix} 0 & 16.38 \\ 22.86 & 6.91 \end{bmatrix} \tag{58}$$

Assuming W_x and W_ϕ are both identity matrices, we have

$$J^T W_x J = \begin{bmatrix} 522.58 & 157.96 \\ 157.96 & 316.05 \end{bmatrix} \tag{59}$$

Let I_1 and I_2 be 0.088 kg-cm^2 and 0.1 kg-cm^2, respectively. Then, the optimal gear ratios can be solved for the above two examples. The resulting structure matrices and their maximum acceleration capacities are given in Table 1. It is clear that for the cases in which the forms of M_r and M_m are compatible, the acceleration capacity can always reach a maximum value and is independent of the gearing configuration.

examples	Structure Matrix (A)	A.C.
1	$\begin{bmatrix} 104.1171 & 0 \\ 0 & -32.6417 \end{bmatrix}$	3.12344
2	$\begin{bmatrix} 103.8970 & 8.9879 \\ 0 & 32.7109 \end{bmatrix}$	3.13062

Table 1: Structures matrices and acceleration capacities

VIII. Summary

We have extended the principle of inertia match from one-D.O.F. system to multi-D.O.F. systems. A methodology for the determination of optimal gear ratios for geared robotic mechanisms has been developed. The methodology is based on the optimization of acceleration capacity at a given posture. We have shown that individual joint-drive manipulators can be designed to achieve an optimum acceleration capacity, although it can not be designed to process a kinematically isotropic property[4]. We have also shown that geared-coupled manipulators can be designed to yield a maximum acceleration capacity, provided the forms of M_r and M_m are compatible with each other. At the optimum design, the mass inertia matrix of input links reflected at the joint-space is equal to that of the major links and the maximum acceleration capacity is independent of the gear train arrangement.

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Appendix

It has been shown [2] that for positive definite matrices X and Y of order n, the following inequality holds

$$\det(X+Y)^{1/n} \geq \det(X)^{1/n} + \det(Y)^{1/n} \quad (A1)$$

Squaring both sides of eq. (A1), we obtain

$$\det(X+Y)^{\frac{2}{n}} \geq \det(X)^{\frac{2}{n}} + \det(Y)^{\frac{2}{n}} + 2\,[\det(X)\det(Y)]^{\frac{1}{n}} \quad (A2)$$

Since it is always true that

$$\det(X)^{2/n} + \det(Y)^{2/n} \geq 2\,\det(X)\det(Y)^{2/n} \quad (A3)$$

It follows, from eqs. (A2) and (A3), that

$$\det(X+Y)^{2/n} \geq 2^2 \det(X)\det(Y)^{2/n} \quad (A4)$$

Taking n/2 power to both sides of eq. (A4), we obtain

$$\det(X+Y) \geq 2^n\,[\det(X)\det(Y)]^{1/2} \quad (A5)$$

Dividing eq. (A5) by $[\det(X+Y)\det(Y)]^{1/2}$, yields

$$\frac{[\det(X)]^{1/2}}{\det(X+Y)} \leq \frac{1}{2^n\,[\det(Y)]^{1/2}} \quad (A6)$$

Replacing X and Y by M_r and M_m in eq. (A6), respectively, and using eq. (15), we obtain

$$\frac{[\det(M_r)]^{1/2}}{\det(M)} \leq \frac{1}{2^n\,[\det(M_r)]^{1/2}} \quad (A7)$$

For $M_r = M_m$, we have

$$\det(M_r + M_m) = \det(2M_r) = 2^n \det(M_r) \quad (A8)$$

Thus, it can be concluded that $M_r = M_m$ is a sufficient condition for the equality sign in eq. (A7) to hold.